Development of an Instrument to Measure Mathematical Sophistication

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Abstract

In this paper we propose a paper-and-pencil instrument to measure the mathematical sophistication of prospective elementary teachers. We call an individual mathematically sophisticated if her mathematical values and ways of knowing are aligned with those of the mathematical community based on nine interwoven traits involving patterns, structures, conjectures, definitions, examples and models, relationships, arguments, language, and notation. In other words, having mathematical sophistication means possessing the avenues of knowing of the mathematical community that allow one to construct mathematics for oneself. We will describe the development of the Mathematical Sophistication Instrument (MSI) and present the results of an initial study of its reliability and validity. We hope the MSI provides the mathematics education community a tool for measuring an important facet of teacher knowledge, and teacher educators a means for assessing pedagogies designed to teach students to think mathematically.

Introduction and Background

What does it mean to know mathematics? This question stands at the center of current battles over mathematics education reform in our schools, the content and design of our national tests, and the ways in which we prepare future elementary and middle grades teachers. The debate over how to best prepare teachers, once centered on the merits of procedural versus conceptual knowledge, has shifted to a more nuanced examination of the latter. Recent research points us to focus on specialized content knowledge related to teaching within the broad category Shulman (1986) identified as *pedagogical content knowledge* (Ball, Hill & Bass, 2005; Ma, 1999; Ball, 2000; Ball, 1993). This works suggests that we must improve "…not just what mathematics teachers know, but how they know it and what they are able to mobilize mathematically in the course of teaching" (Ball, 2000, p. 243).

Through study of their own work as elementary teachers, both Ball (1993) and Lampert (2000) have rendered detailed accounts of the mathematical work of teaching. More recently, Ball and her colleagues (e.g., Ball, Hill & Bass, 2005) have set out to identify and measure a *mathematical knowledge for teaching* based on the mathematical work of teaching. They assert that teachers must understand the mathematical definitions, representations, examples and notations that are most powerful in supporting children's understanding; they must hear the mathematical thinking of children and guide and extend that thinking; they need to recognize the nature of children's alternate conceptions and help them to create counterexamples and arguments. In their study of first and third grade classrooms, Ball, Hill & Bass (2005) demonstrated that teachers' mathematical knowledge for teaching positively predicted gains in mathematical achievement of their students.

In a recent paper (Seaman & Szydlik, 2007) we proposed yet another possible answer to the question of what it means to know mathematics. We termed the construct *mathematical sophistication* and argued that it is helpful in explaining why some preservice teachers fail to learn mathematics. Specifically, we showed that prospective elementary teachers in our study could not use a teacher resource to refresh fundamental mathematical ideas, and we asserted that this failure was due, in large part, to their lack of mathematical sophistication. While this construct is informed by our own work as mathematicians and our observations and reading of the work of practicing research mathematicians, it seems intimately intertwined with the mathematical work of teachers described by Ma and Ball.

While many researchers have measured mathematics attitudes, beliefs, procedural skills, or conceptual knowledge, few have sought to quantify mathematical *behavior*. Perhaps the most closely related work was done by Schoenfeld (1992), in which he contrasts the differing behavior of college students and mathematics faculty in problem solving. He found that students typically, after reading the problem, made a quick decision on what approach to take, and stuck with that approach even if it did not lead to a successful solution. In contrast, he found that mathematics faculty members spent significant time in understanding the problem, analyzing, and exploring. He writes that "for the most part, students were unaware of, or failed to use, the executive skills demonstrated by the expert" (p. 356). What Schoenfeld called "expert executive skills" is part of what we termed "mathematical sophistication," or the ability to engage in mathematically sophisticated behaviors.

In another relevant study, Hill, Schilling and Ball (2004) wrote and tested a multiple choice survey to measure the mathematical pedagogical content knowledge of K-12 mathematics teachers. But this survey did not look specifically at the mathematical behaviors of the teachers; instead it focused, among other things, on teachers' abilities to recognize children's mistakes and to assess the validity of mathematical procedures. In this paper, we describe our work to develop an instrument to measure mathematical behavior.

Mathematical Sophistication Framework

We use *mathematical sophistication* to describe internalization of the values and behaviors of the mathematical community. In other words, a mathematically sophisticated individual has taken *as her own* the values and ways of knowing of that community. The difference between a sophisticated mathematics student and a naive one lies in her beliefs about the nature of mathematical behavior, her values concerning what it means to know mathematics, her avenues of experiencing mathematical objects, and her distinctions about language and notation.

In Seaman & Szydlik (2007) we proposed the following list of traits that indicate mathematical sophistication. We assert that these traits are not disjoint categories, but are interrelated and used in concert by mathematicians as they solve problems and create mathematics.

1) Mathematicians seek to understand patterns. “Seeing and revealing hidden patterns is what mathematicians do best” (Steen, 1990, p. 1).
2) Mathematicians make analogies by finding the same essential structure in seemingly different mathematical objects. “Mathematics is the art of giving the same name to different things” (Poincaré as found in O’Connor & Robertson, 2003).
3) Mathematicians make and test conjectures about mathematical objects and structures. “When you try to prove a theorem, you don’t just list the hypothesis, and then start to reason. What you do is trial and error, experimentation, guesswork” (Halmos, 1985, p. 321).
4) Mathematicians create mental (and physical) models, and examples and non-examples of mathematical objects. This is the way we come to create and understand our definitions, and thus understand our mathematical objects. “A good stock of examples, as large as possible, is indispensable for a thorough

understanding of any concept, and when I want to learn something new, I make it my first job to build one" (Halmos, as found in Gallian, 1998, p. 40).

5) Mathematicians value and use precise definitions of objects (Tall, 1992). Our definitions provide us necessary and sufficient criteria for classifying objects and making arguments. "The mathematician is not concerned with the current meaning of his technical terms… The mathematical definition *creates* the mathematical meaning" (Polya, 1957, p. 86).

6) Mathematicians value an understanding of why relationships make sense. "Mathematicians do not study object, but relations among objects; they are indifferent to the replacement of objects by others as long as relations do not change. Matter is not important, only form interests them." (Poincaré as found in Gallian, 1998, p. 115).

7) Mathematicians value and use logical arguments and counterexamples as our sources of conviction (Tall, 1992). These help us to understand relationships among mathematical objects and provide us autonomy. "Proof is the idol before whom the pure mathematician tortures himself" (Eddington, 1928, p. 337).

8) Mathematicians value precise language and have fine distinctions about language. We need these to communicate assertions and to make and evaluate arguments. For example, we carefully distinguish between "and" and "or," "there is something, such that for all" and "for all, there is something such that," "at most" and "at least," necessary and sufficient conditions, and converse and contrapositive forms, to name just a few. "Ordinary language is totally unsuited for expressing what physics really asserts, since the words of everyday life are not

sufficiently abstract. Only mathematics and mathematical logic can say as little as the physicist means to say" (Russell, 1931, p. 82).

9) Mathematicians *value* symbolic representations of, and notation for, objects and ideas because these help us to organize our own thinking and to communicate meaning to others. "In symbols one observes an advantage in discovery which is greatest when they express the exact nature of a thing briefly and, as it were, picture it; then indeed the labor of thought is wonderfully diminished" (Gottfried Wilhelm Leibniz as found in Simmons, 1992, p.156).

We stress that having mathematical sophistication does not imply an understanding of any specific definition, mathematical object, or procedure. Rather, it means possessing the *avenues of knowing* of the mathematical community that allow one to construct mathematics for oneself.

Methodology

We had several criteria in mind when we developed the Mathematical Sophistication Instrument (MSI). The first was to require only elementary mathematics content knowledge. On the MSI we often define new mathematical objects and provide novel definitions and examples of those objects, and we ask students to reason about them. That way, students cannot rely upon previously learned mathematics, but must demonstrate an ability to learn new mathematics. Second, we intended that all nine sophistication traits be reasonably represented on the MSI. Third, in order to ease administration and scoring, we decided upon a multiple choice, paper-and pencil test that could be completed during the course of a standard class period (45 minutes). Though we developed two parallel forms of the MSI to serve as pre-and post-tests, in this report we focus on just one form. We now describe the development process.

In the summer of 2007, a panel of seven expert learners of mathematics (mathematics professors) evaluated an initial version of the instrument and classified each item according to sophistication traits and to the level of sophistication required to answer each item correctly. Then six elementary education majors (who were also mathematics minors) completed the instrument and each participated in an informal interview during which the student explained his or her interpretations of the items and response options and his or her reasons for selecting answers. We used that data to revise some items and replace others.

In fall (2007) we administered the revised MSI to a large sample of elementary education majors in their mathematics content courses. Twelve students, four scoring in the bottom quartile, four scoring the middle half, and four scoring in the upper quartile participated in semi-structured interviews. Those students solved a subset of the items and described their interpretations and thinking about the items and response options. The purpose of the interviews was to determine the extent to which sophisticated thinking produced correct responses on the MSI and unsophisticated thinking produced incorrect responses. Data from the pilot test and interviews was used to change or delete items which were problematic or had negative item-test correlations.

We administered the resulting version of the MSI to 56 elementary education students in their mathematics content courses in the fall of 2008, and we asked the three faculty members teaching those courses to rate the mathematical sophistication of each of their own student participants on a five-point linear scale. Below we report the results.

Results

This version of the instrument consists of 25 items. Students earned one point per item for the most sophisticated response (according to the mathematician panel) and no points for any other option. Thus scores could potentially range from 0 to 25. Students were not given any incentives to do their best on the instrument, and so some students may not have scored as high as their potential. The distribution of 56 elementary education students' scores from fall 2008 is shown in Table 1. The mean score was 11.7, the median was 11, and the standard deviation was 4.65.

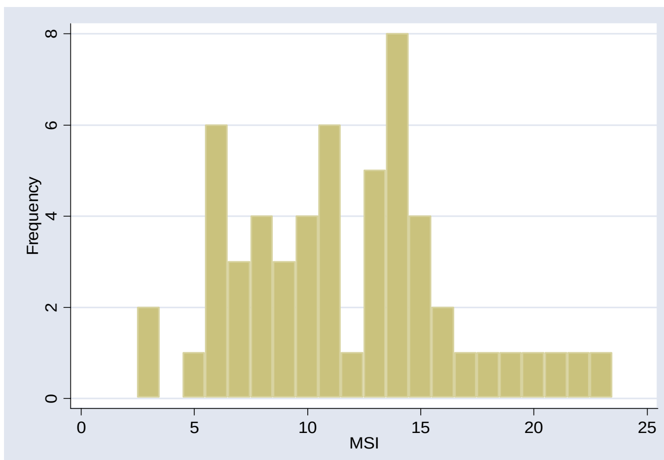


Table 1. MSI Scores for Fall 2008, N = 56.

On this instrument each sophistication trait is represented as a primary theme at least twice. For example, MSI Sample Item 1 (see Figure 1) was classified by the mathematicians to primarily measure distinctions about language, and in particular the meaning of "at most." The item may measure the use of a logical argument as a secondary trait. It has an item-test correlation of 0.43 indicating that students who selected the most sophisticated answer were also likely to score well on the instrument as a whole. The most sophistication response is bolded and the percentage of participants who selected each response option appears in parentheses.

MSI Sample Item 1: Consider the following statement:

There are at most ten people in the swimming pool.

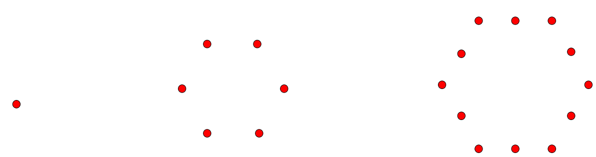
Assuming the above statement is true, which of the following statements must also be true?

a) There are ten people in the swimming pool. (12.5%)
b) There is at least one person in the swimming pool. (21.4%)
c) Both of the above statements must be true. (12.5 %)
d) None of the above statements must be true. (53.6%)

Figure 1. MSI Sample Item 1.

MSI Sample Item 2 (see Figure 2) is designed to measure a student's ability to recognize and continue a pattern. It also measures the ability to make sense of a provided definition (of "hexagonal frame number"), notation, and language. The item-test correlation for this item is 0.45. On the instrument, it is followed by an item that asks the students to find HF_{101}.

MMSI Sample Item 2: The numbers 1, 6 and 12 are called **hexagonal frame numbers** because one dot, six dots, and twelve dots can each be arranged in the shape of a hexagon frame as follows:



one dot $HF_1 = 1$ | six dots $HF_2 = 6$ | twelve dots $HF_3 = 12$

We say that 1 is the first hexagonal frame number (HF_1). The second hexagonal frame number (HF_2) is 6, and so on. What is the fourth hexagonal frame number (HF_4)?

a) $HF_4 = 18$ (58.9%)
b) $HF_4 = 19$ (12.5%)
c) $HF_4 = 24$ (26.8%)
d) None of the above (1.79%)

Figure 2. MSI Sample Item 2.

While the majority of items require students to *do* mathematics, two items ask that students express how they likely would respond to a hypothetical situation. Consider, for example, MSI Sample Item 3 (see Figure 3) with an item-test correlation of 0.47. This item was crafted to test whether students *value* exploring conjectures and understanding why relationships make sense.

> MSI Sample Item 3: You are a student learning about using lines to model data and, after the lesson, a student raises her hand and makes a guess about how other types of functions could be used to model data. Which option best reflects your view?
>
> a) I would probably just want to know whether she was correct or not. (16.1%)
> **b) I would probably want to spend time exploring her guess myself. (48.2%)**
> c) I would probably prefer to focus only on the material that was part of the real lesson. (7.14%)
> d) I would probably want the instructor to figure it out and explain it to me. (28.6%)

Figure 3. MSI Sample Item 3.

All 25 items have a positive (typically between 0.30 and 0.50) item-test correlation indicating reasonable internal consistency. Furthermore the instrument as a whole has a Cronbach alpha of 0.7755 suggesting that it reliably measures a single construct. In order to show that this construct is *mathematical sophistication*, we assessed the validity of the instrument in two ways.

In the development stage, seven mathematicians identified all the sophistication traits each item satisfied. Items on which the experts disagreed were discarded or re-written until consensus was reached. In other words, our panel of experts agreed that all the items measured at least one trait of mathematical sophistication.

In a second test of validity, the three instructors of the students who completed the instrument rated each of their students on a five-point linear scale, where a 5 was defined to indicate a highly sophisticated elementary education student. Instructors did not rate those students for which they had insufficient evidence to judge mathematical sophistication. Forty-three of the 56 students were rated.

Three students were rated at Level 1 (highly unsophisticated); eleven students were rated at Level 2 (fairly unsophisticated); 21 students were rated at Level 3 (neutral sophistication); seven were rated at Level 4 (fairly sophisticated) and one student was rated at Level 5 (highly sophisticated). Table 2 shows the mean scores on the instrument for the students at each level. Students rated by their instructor at Level 2 scored an average of 4.4 points higher than those rated at Level 1; students rated at Level 3 scored an average of 1.37 points higher than those at Level 2; and students rated at Level 4 scored an average of 3.76 points higher than those at Level 3. All differences in means between levels are significant at the 0.05 level with the exception of the difference between the means from Levels 2 and 3. The Level 5 student scored a 15 on the instrument.

Student Rating by instructor	Number in Category	Mean Score on the MSI
1	3	6.33
2	11	10.73
3	21	12.10
4	7	15.86

Table 2. Instructor Ratings of Students' Mathematical Sophistication and Mean Score of each Rating Category on the MSI. N = 43.

Conclusion

The preliminary results suggest that the MSI is a reliable and valid measure of the mathematical sophistication of prospective elementary teachers. As such, the instrument promises to provide the mathematics education community a tool for measuring an important facet of teacher knowledge, and provide teacher educators a means for assessing pedagogies designed to teach prospective teachers to think mathematically.

Future development of the instrument could include validating its use in measuring the mathematical sophistication of practicing teachers, as well as mathematics students in general, which would allow the instrument to become an assessment tool for professional development and college mathematics programs. In the fall of 2009 we intend a larger scale assessment of the instrument involving a wider variety of mathematics students from several campuses. College mathematics instructors and teacher educators who are interested being a part of this planned assessment are encouraged to contact the authors for more information.

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